GRAPH THEROY

Class: I M.Sc Maths

Subject: Graph Theory

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Defintiion:

A graph G consists a pair of $\left(V(G), X(G)\right)$, where $V(G)$ is a non-empty finite set whose elements are called **points or vertices** and $X(G)$ is another set of unordered pairs of distinct elements of *V(G)*. The elements of *X(G)* are called **lines or edges** of the graph.

If $x = \{u, v\} \epsilon X$ then the line *x* said to join *u* and *v*. The points *u* and *v* are said to **adjacent** if $x = uv$. The points *u* and the line *x* are **incident** with each other.

If two distinct lines *x* and *y* are incident with a common point then they are called **adjacent lines.**

A graph with *p* points and *q* lines is called a $(\boldsymbol{p}, \boldsymbol{q})$ **graph**.

NOTE:

Denote $V(G), = V\ X(G) = X$

Directed Graph

A **directed graph** D is an ordered triple $(V(D), A(D), I_D)$ where $V(D)$ is a nonempty set called the set of vertices of *D*, A(D) is a set disjoint from $V(D)$ called the set of arcs of *D* and I_D is an incidence map that associates with each arc of D an ordered pair of vertices of D. If *a* is an **arc of D** and $I_D(a) = (u, v)$, *u* is called the **tail of a** and *v* is the **head of a**. The arc *a* is said to **join *v* with *u***. *u* and *v* are called the **ends of *a***. A directed graph is also called a **digraph**.

Underlying Graph and Orientation

With each digraph D there is an associate graph G (written G(D) when reference to D is needed) on the same vertex set as follows: Corresponding to each arc of D, there is an edge of G with the same ends. This graph G is called the **underlying graph of the digraph** D:

Conversely, given any graph G there is a digraph from G by specifying for each edge of G an order of its ends. Such a specification is called an **orientation** of G:

A digraph and its underlying graph are shown in Fig. 1.

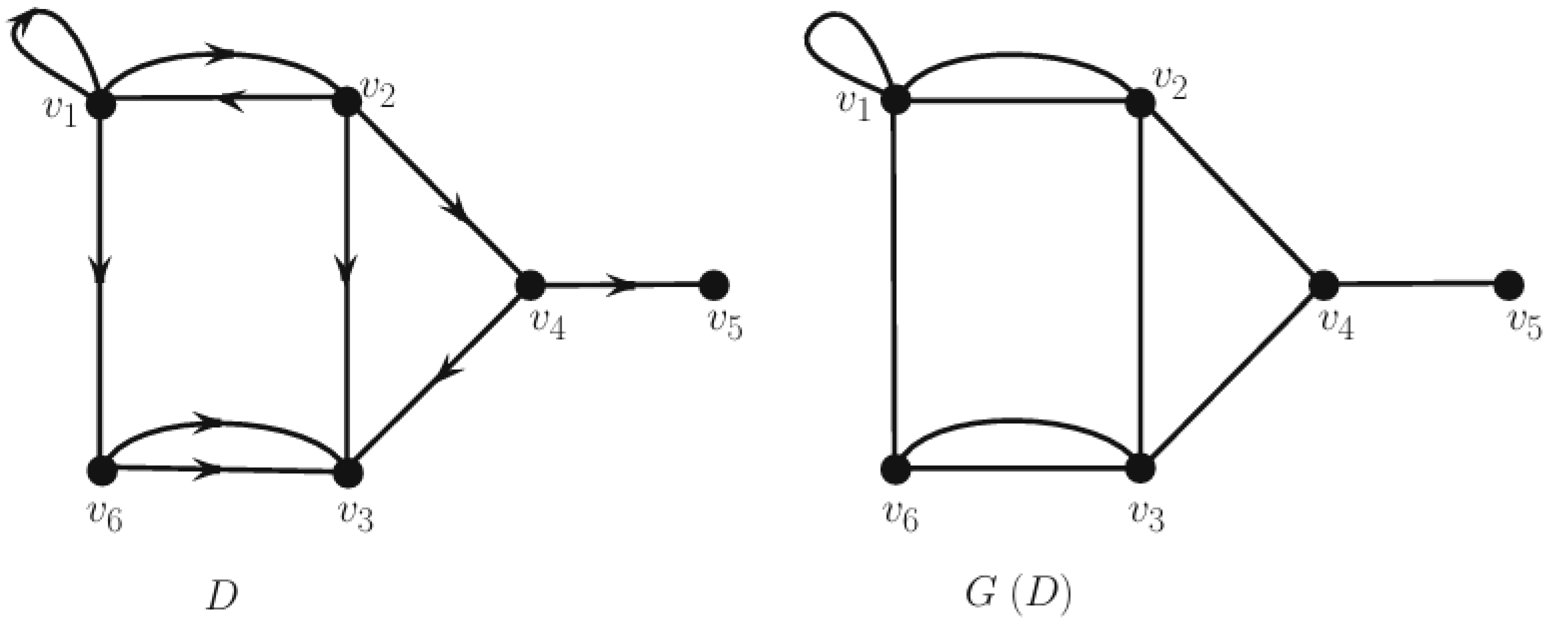

Fig. 1 Digraph D and its underlying graph G(D)

Definition

If $a = (u, v)$ is an arc of D, *a* is said to be incident out of *u* and incident into *v*. *v* is called an **outneighbor** of *u* and *u* is called an **inneighbor** of *v*. $N_D{}^+$denotes the set of outneighbors of *u* in D. Similarly, $N_D{}^-$ denotes the set of inneighbors of *u* in D. When no explicit reference to D is needed, denote these sets by $N_D{}^+$ and $N_D{}^-$ respectively. An **arc *a* is incident with *u*** if it is either incident into or incident out of *u*. An arc having the same ends is called a **loop** of D. The number of arcs incident out of a vertex *v* is the **outdegree of *v*** and is denoted by $d_D{}^+(v)$ or $d^+(v)$. The number of arcs incident into *v* is its **indegree** and is denoted by $d_D{}^-(v)$ or $d^-(v)'$

For the digraph D of Fig.1, $d^+(v_1) = 3,\ d^+(v_2) = 3,\ d^+(v_3) = 0,\ d^+(v_4) = 2,$ $d^+(v_5) = 0,\ d^+(v_6) = 2,\ d^-(v_1) = 2,\ d^-(v_2) = 1,\ d^-(v_3) = 4,\ d^-\left(v_{4,}\right) = 1,\ d^-\,(v_5) = 1,$ $d^-(v_6) = 1$. The loop at v_1 contributes 1 each to $d^+(v_1)$ and $d^-(v_1)$

The **degree $d_D(v)$ of a vertex v** of a digraph D is the degree of v in G(D). Thus, d(v) $= d^+(v) + d^-(v)$. As each arc of a digraph contributes 1 to the sum of the outdegrees and 1 to the sum of indegrees,

$\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v) = m(D)$ where $m(D)$ is the number of arcs of D.

A vertex of D is **isolated** if its degree is 0; it is **pendant** if its degree is 1. Thus, for a pendant vertex v either $d^+(v) = 1$ and $d^+(v) = 0$ or $d^+(v)$ =0 and $d^+(v)$ =1.

Definitions

1. A digraph is a D' **subdigraph** of a digraph D if $V(D') \subseteq V(D), A(D') \subseteq A(D)$ and $I_{D'}$ is the restriction of I_D to $A(D')$.
2. A **directed walk** joining the vertex v_0 to the vertex v_k in D is an alternating sequence $W = v_0\ v_0 a_1 v_1 a_2 v_2 \ldots a_k v_k, 1 \leq i \leq k$ with a_i incident out of v_{i-1} and incident into v_i.

3. A **vertex *v* is reachable from a vertex *u*** of D if there is a directed path in D from u to v.
4. **Two vertices of D are diconnected** if each is reachable from the other in D. Clearly, diconnection is an equivalence relation on the vertex set of D, and if the equivalence classes are $V_1, V_2, \dots V_\omega$, the subdigraphs of D induced by $V_1, V_2, \dots V_\omega$ are called the dicomponents of D.
5. **A digraph is diconnected** (also called strongly-connected) if it has exactly one dicomponent. A diconnected digraph is also called a **strong digraph**.
6. A digraph is **strict** if its underlying graph is simple. A digraph D is **symmetric** if, whenever (u, v) is an arc of D, then (v, u) is also an arc of D.

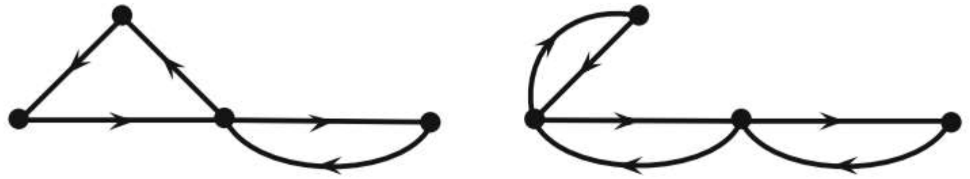

Fig. 2. A strong digraph (left) and symmetric digraph (right)

Tournaments

A digraph D is a **tournament** if its underlying graph is a complete graph. Thus, in a tournament, for every pair of distinct vertices *u* and *v* either (u, v) or (v, u), but not both is an arc of D.

Example

Suppose there are *n* players in a tournament and that every player is to play against every other player. The results of such a tournament can be represented by a tournament on *n* vertices, where the vertices represent the n players and an arc (u, v) represents the victory of player u over player v.

Suppose the players of a tournament have to be ranked. The corresponding digraph T, a tournament, could be used for such a ranking.

The ranking of the vertices of T is as follows:

One way of doing it is by looking at the sequence of outdegrees of T, because ${d_T}^+(v)$ stands for the number of players defeated by the player v.

Another way of doing it is by finding a directed Hamilton path, that is, a spanning directed path in T. One could rank the players as per the sequence of this path so that each player defeats his or her successor.

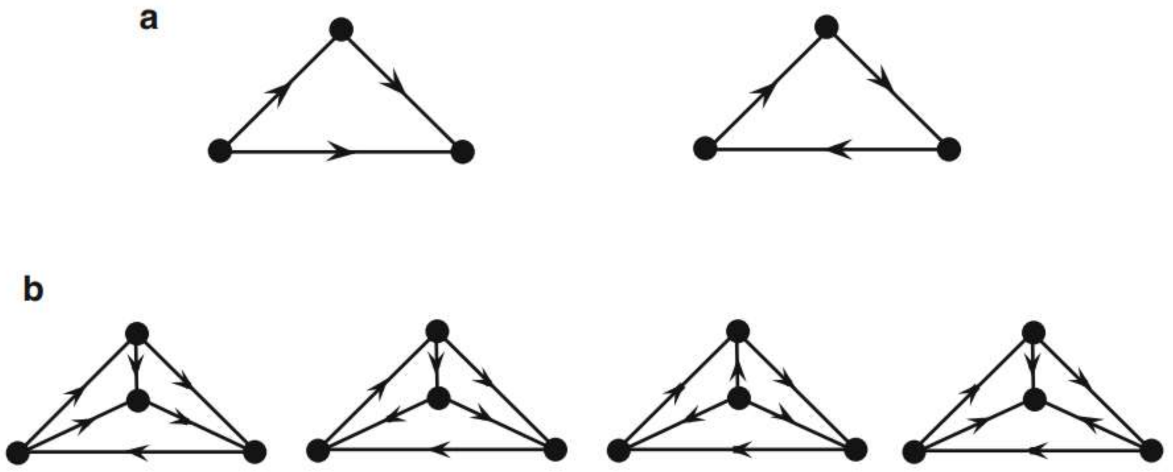

Fig.3. Tournaments on (a) three and (b) four vertices

Theorem 1

Every tournament contains a directed Hamilton path

Theorem 2 (Moon Theorem)

Every vertex of a diconnected tournament T on n vertices with $n \geq 3$ is contained in a directed k-cycle, $3 \leq k \leq n$ (T is then said to be vertex-pancylic)

Remark

Theorem 2 shows, in particular, that every diconnected tournament is Hamiltonian; that is, it contains a directed spanning cycle.


thank you